Primary Pre-service Teachers' Noticing of Structural Thinking in Mathematics

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Stephens (2008) described structural thinking as an awareness of the way different occurrences of a mathematical property develop into correct generalisations. Many primary pre-service teachers (PSTs) graduate from their tertiary studies without the ability to notice structural thinking. In this study, two primary PSTs learned to notice structural thinking in three cycles of professional learning workshops; teaching mathematics; and, interviews. Here, we report on how the PSTs attended to structural thinking in their classroom communication and their awareness of doing so in the workshops and interviews. Results indicate the PSTs improved their noticing of structural thinking.

A lack of mathematical content and pedagogical knowledge is highlighted as a reason teachers do not notice students' mathematical thinking (Ivars, Fernández-Verdú, Llinares, & Choy, 2018). Schoenfeld's (1992) metacognitive perspective of mathematical thinking includes structural thinking as it involves attending to one's thinking when doing mathematics (Mason, 2004). Structural thinking encompasses mathematical structure, referred to here as *structure*; it involves the learner knowing what procedures to use while understanding the mathematical concepts behind the procedures (Mason, Stephens, & Watson, 2009). Mason et al. (2009) considered structure to be the bridge between procedural and conceptual understanding (Hiebert & Lefevre, 1986). PSTs' awareness of structure could build this bridge; however, this awareness must be identified when teaching. Effective teaching requires teachers to have an awareness of their own and their students' structural thinking.

Cavanagh (2006) stated in his report on curriculum changes designed to improve students' mathematical thinking that it is imperative teachers understand structural thinking. Teachers also need to notice their own and their students' structural thinking, as it involves applying their mathematical and pedagogical knowledge to teaching and learning mathematics. Mason (2002) introduced the concept of *noticing* as an awareness of what one attends to when acting-in-the-moment of teaching. Subsequently, continued research in noticing recognises this concept as important in mathematics teacher education. Jacobs, Lamb, and Philip (2010) developed a framework for teachers to notice students' mathematical thinking. Van Es (2011) proposed noticing as a requirement for mathematics teachers and Anthony, Hunter, and Hunter (2015) used noticing to support PST mathematics education. This paper supports Ivars et al. (2018) claim that PSTs can learn to notice mathematical thinking, and reports on how two primary PSTs learned to notice structural thinking.

Literature Review

Mason et al. (2009) described how structure supports teachers to recognise deep thinking and understanding of mathematical concepts and, when combined with the mastering of procedures, it is essential to make sense of mathematics. The mastery of procedures alone is of little use when the procedures increase, and memory becomes overloaded; having an awareness of structure supports a shift in the learning from relying purely on memorising of procedures.

2019. In G. Hine, S. Blackley, & A. Cooke (Eds.). Mathematics Education Research: Impacting Practice *(Proceedings of the 42nd annual conference of the Mathematics Education Research Group of Australasia)* pp. 324-331. Perth: MERGA.

Mason (2004) identified what learners attend to when solving mathematical problems. He argued that "structures of attention" need to be identified and promoted by the teacher as mathematical thinking skills that can support structural thinking. Four components of structure, given here, build on Mason's structures of attention:

***C**onnections* with prior and future mathematical learning and known mathematical concepts (Richland, Stigler, & Holyoak, 2012).

***R**ecognising of patterns and relationships* through identifying and reproducing content and concepts (Mulligan & Mitchelmore, 2009).

***I**dentifying similarities and differences*, including equivalences in all content and concepts (Vale, McAndrew, & Krishnan, 2011).

***G**eneralising and reasoning* of expressions, relationships between properties, explaining, and justifying conclusions (Watson & Mason, 2005).

These four components, given the acronym of the CRIG framework (Gronow, 2015), were developed as a workable teaching framework to identify teachers' understanding and use of mathematical structure.

This study examines two PSTs' communication in their teaching practice as they learned to notice structural thinking, furthering the first author's investigation of teachers' understanding and use of mathematical structure (Gronow, 2015).

Mason et al. (2009) stated that structure could not be taught, so identifying PSTs' awareness of structure is difficult. In this study, the PSTs learned the CRIG framework as a means of attending to structure. As PSTs attend to structure, they also develop their metacognitive awareness of noticing structural thinking in mathematics.

Research Questions

The research questions are:

1. How useful is the CRIG framework in helping PSTs to notice structural thinking in their teaching?
2. How do PSTs attend to the CRIG framework to notice structural thinking when teaching mathematics?

Method

Context and Participants

Two final year Bachelor of Education/Bachelor of Arts degree primary education PSTs at a Sydney university, referred to as Ms S and Ms N, volunteered to participate in this study, which took place during their three-month professional experience placement. Each PST was allocated to a primary school in metropolitan Sydney and taught a Year 1 class (6-7-year-old students). The PSTs had completed mathematics education units of their undergraduate degree that contained the concept of structure, and they were exposed to aspects of structure through the New South Wales' K-10 mathematics syllabus; however, they had no prior knowledge of the CRIG teaching framework.

Study Design, Instruments and Data Collection

This qualitative interpretive study followed the PSTs over three cycles of (1) professional learning workshop; (2) planning and teaching a mathematics lesson, that was videoed; and (3) post-lesson audio-recorded interview.

Professional learning workshops: PSTs participated in three workshops to learn about structural thinking. The workshops introduced the CRIG framework to the PSTs, and the prepared lesson plans using the CRIG framework (Workshop 1 and 2) were discussed. PSTs

viewed and reflected on short video segments from each PST's mathematics lesson (Workshop 2 and 3) in order to move beyond guided unpacking so that they could begin to notice their structural thinking.

Mathematics lessons: Each PST taught three lessons that were video-recorded. The lesson topics were determined by the professional experience supervising teachers' mathematics program. The lesson topics for Ms S were: Lesson 1 Multiplication using arrays; Lesson 2 Estimating length; Lesson 3 Division by sharing; and for Ms N: Lesson 1 Developing patterns; Lesson 2 Adding 2-digit numbers; Lesson 3 Classifying three-dimensional (3D) objects**.** After each lesson, the PSTs submitted their lesson plan that described how they included CRIG framework.

Interviews: Each PST was interviewed immediately after their respective mathematics lesson; the interview was audio-recorded and took place in a quiet space adjacent to the classroom. The interview provided an opportunity for PSTs to reflect on their lesson and opportunities to attend to their own structural thinking and become aware of when they used the CRIG framework.

Analysis

Data from the audio-recordings of the professional learning workshops and interviews were all transcribed to a word document and with the lesson plans were uploaded to NVivo 12 (QSR International, 2017), then coded for each component of the CRIG framework.

The mathematics lesson videos were viewed by the researcher, and key terms and phrases that identified the CRIG framework and amount of time they were attended to were recorded and coded to each component of the CRIG framework. Exemplars of the PSTs attending to the CRIG framework to notice structural thinking were transcribed in more detail.

As a qualitative interpretive study, the variety of data collected was analysed for evidence of the PSTs' noticing of structural thinking through the PSTs attending to and awareness of the CRIG framework. Each instrument provided an alternative perspective of the data; for example, the researcher recorded the PSTs attending to CRIG framework in the mathematics lessons and the PSTs gave personal reflections on their awareness of the attending to the CRIG framework to notice structural thinking in the interviews and professional learning workshops. The videoed mathematics lessons acted as the main source of evidence used to analyse the use of CRIG framework; the lesson plans and interviews provide further evidence of where the PSTs applied the CRIG framework in their teaching. Data from the professional learning workshops presented the PSTs' developing views on noticing of structural thinking.

Results

This section presents a summary of the results from the three cycles of professional learning workshops, mathematics lessons, and interviews. Exemplars are given of the PSTs attending to and awareness of the CRIG framework from the workshops, the mathematics lessons and interviews. Additionally, the percentage of time the PSTs attended to the CRIG framework recorded from the mathematics lessons and the frequency of PSTs' references as awareness of the CRIG framework from the interviews are given.

Cycle 1

Workshop 1: This workshop began with an introduction to structural thinking and the CRIG framework. Both PSTs demonstrated structural thinking as described by Stephens (2008) when solving arithmetic number sentence problems using mathematical relationships

rather than mental calculations. During the workshop, Ms N stated that noticing students' structural thinking is "understanding how students process mathematics".

Lesson 1: During this lesson on multiplication using arrays, Ms S attended to the CRIG framework, but rarely referred to a component by its name. *Similarities and differences* (43%), followed by *Connections* (25%), appeared most often as Ms S was comparing multiplication to addition. *Generalising and reasoning* (14%) and *Recognising patterns* (8%) were used when explaining that multiplication could be represented by repeated addition, grouping and arrays.

In this lesson, Ms S's questions invited the students to make connections to what they already knew about multiplication. She asked the class about the meaning of seven times seven, a student was confused and responded with "Seven times seven is probably fourteen". This encouraged Ms S to discuss the similarities and differences between multiplication and addition. She asked another student what seven times seven meant, this student connected multiplication to repeated addition, stating that "it is seven added together seven times". Ms S furthered this response by developing a pattern, through drawing a diagram she showed how multiplication can be displayed as equal groups. Consequentially, she generalised that multiplication is different to addition, and by showing an array pattern of seven rows of seven was able to demonstrate multiplicative structure.

In Ms N's first lesson on patterns she attended to CRIG framework through *Recognising patterns* (59%), and *Identifying similarities and differences* (16%), fewer references to *Connections* (7%) and *Generalising and reasoning* (7%) were made. The lesson topic of patterns justifies the high occurrence of attending to the *Recognising patterns* component; however, attending to other components was in how the topic connected to the students' world, such as, describing a decreasing pattern as "When you are going to lunch you are going down the stairs". Students were encouraged to recognise different increasing and decreasing patterns and then to generalise a pattern. Ms N noticed structural thinking in how students had difficulty generalising the pattern when the blocks were arranged horizontally, but by rearranging the blocks vertically she noticed how students were able generate the pattern.

Interview 1: Ms S's awareness of the CRIG framework was identified in the seven references to *Connections* and five to *Generalising and reasoning* in her interview. Ms. N's awareness of the CRIG framework was noted in 11 references to *Recognising patterns* and four to *Identifying similarities and differences* with two references each to *Connections* and *Generalising and reasoning*. Both PSTs made observations about their attending to the CRIG framework when teaching, Ms S stated she was not consciously aware of it when teaching, while Ms N felt that it was useful.

> Ms S: I'll be honest, when I'm teaching it. I don't think about these CRIG components.
>
> Ms N: They could recognise the patterns themselves .... and then they could start to identify similarities and differences between patterns that they had made and that their partner has made.

Cycle 2

Workshop 2: PSTs viewed a segment of Ms N's lesson on patterns. Ms N's awareness of the CRIG framework was that it is a teaching skill that teachers just do without knowing.

> Ms N: They could become a natural part where you don't actually think about using them, but they just are implicit in your teaching.

Lesson 2: Ms S's lesson required students to estimate and measure lengths using informal units. The percentage of time Ms S attended to the CRIG framework were *Generalising and reasoning* (48%) which was almost double that of *Identifying similarities and differences* (24%) and *Connections* (20%). The lesson advanced students' generalising

ability by choosing appropriate informal units and then to make reasonable estimations. Ms S's attention to connections was through asking students what they know about length and similarities and differences found in the informal units of measurement they chose.

In her lesson, Ms N used the jump strategy on an open number line to demonstrate addition of two-digit numbers. This lesson was dominated by Ms N's attending to *Recognising patterns* (38%) and *Connections* (26%), and the class discussion began by connecting knowledge of two-digit numbers to one-digit numbers. Ms N decomposed numbers to recognise the number patterns found when using partitioning of numbers to generalise the structure of two-digit numbers.

Interview 2: Ms S's awareness of the CRIG framework appeared fewer times overall in this interview, with only four references of *Generalising and reasoning*, two to *Connections* and one to both *Recognising patterns* and *Identifying similarities and differences*. Ms S again stated she had not noticed the framework when teaching.

> Ms S: When I was doing the lesson, I think it was all about explaining, explaining, explaining what to do rather than, at that time, that I can identify the CRIG components.

Ms N's awareness of the CRIG framework, was identified in her six references to *Connections*, five to *Recognising patterns*, four to *Generalising and reasoning* and two to *Identifying similarities and differences*. Ms N noticed structure when declaring that students did not demonstrate their structural understanding of two-digit numbers when using mental calculations.

> Ms N: Students, instead of using the number line strategy, were actually jumping straight to the use of mental strategies without using the number line. They weren't able to show me they understood.

Cycle 3

Workshop 3: Ms S's awareness of the usefulness of the CRIG framework developed during the study and was noted in this reference to connections.

> Ms S: When I think of connections, the only thing I think of is prior knowledge. … But when you teach your lesson, you always forget to say. Oh, where else do you see it? That's the spot, relate it back to something.

Ms N's awareness of the CRIG framework was noticed in how it helped her to examine her teaching.

> Ms N: A big learning for me this professional experience is that I'm teaching…I have to strip it right back. …. But really, you want to process one thing at a time.

Lesson 3: Ms S's third lesson attended to the CRIG framework components of *Connections* (38%) with *Recognising patterns* (22%) and *Identifying similarities and differences* (22%) with less attention to *Generalising and reasoning* (4%). The significant use of connections relates to Ms S's lesson topic of division, which she connected to her first lesson topic on multiplication, and she also connected mathematical numerals and symbols to language familiar to the students. On the whiteboard Ms S wrote "$12 \div 4 =$ " and above the 12 she wrote "total", above the division sign she wrote "share" and next to the four she wrote "how many groups?". Ms S was identifying the similarities and differences between these two operations and used patterns to model division when demonstrating how 12 dots are evenly distributed among the four circles, to show three dots in each circle.

In Ms N's categorising of 3D objects lesson, she used *Identifying similarities and differences* (42%) to identify flat and curved surfaces of 3D objects. *Connections* (14%) was exhibited when showing familiar 3D objects to the students, and *Generalising and reasoning* (11%) was identified when discussion with the students involved how two hoops can be arranged to sort the 3D objects (all flat sides, all curved sides, both curved and flat sides).

Interview 3: In this interview Ms S's awareness of the CRIG framework occurred more than previously with ten references to *Identifying similarities and differences*, nine to *Recognising patterns*, four to *Generalising and reasoning* and three to *Connections*. Ms S gave an insightful understanding of pattern recognition as the *"*transferring of concepts into different concepts", which identifies the nature of structure in mathematical relationships. Ms S also described the CRIG framework as helping students think about the relationship between multiplication and division.

> Ms S: the CRIG framework did, it really triggered the students into thinking about the relationship.

In this third interview, Ms S's awareness of the CRIG framework was noted when she stated how she kept it in mind when teaching; she was aware of it when teaching and reminded herself to use it.

> Ms S: All of these CRIG components fit somehow into my teaching. If I really had to sit down and think about it. … I think yeah, in the activity they did do that. So, I kept this in mind. I kept prompting myself how can I push these components into the teaching? Yeah, and it worked.

Ms N made fewer references to the CRIG framework in this interview than in the previous two. Seven references to *Generalising and reasoning*, five to *Identifying similarities and differences*, three to *Connections* and one to *recognising patterns*. Although, while not referring to the individual components of the CRIG framework, Ms N reported that the CRIG framework assisted her to assess students' thinking and learning as it happens in the moment of the lesson.

> Ms N: It helped me, it has really drilled home that the most amount of thinking and learning is happening in the moment, in the middle of that lesson, not at the end when they are assessed.

Discussion

The first research question considers if learning about the CRIG framework is useful in helping PSTs to notice structural thinking. The results showed the PSTs did attend to the CRIG framework in their teaching and were aware of it in the workshops and interviews. Attending to the CRIG framework and awareness of doing so proved to be useful in helping the PSTs to notice structural thinking when teaching.

PSTs acknowledged that inexperience hampered their ability to attend to the CRIG framework when teaching. Ms S said she did not use the CRIG framework, although analysis of her lessons demonstrated regular references to the framework. Ms N stated that she felt that the CRIG framework would become familiar in her pedagogy over time, reflecting research that states expert teachers notice more than novices (Mason, 2002). These experiences are reinforced by occasions when the PSTs missed opportunities to notice structural thinking and the PSTs' attention was not on the students' learning, confirming Star and Strickland's (2008) observation that novice teachers do not necessarily focus on their pedagogical approach or awareness of student learning. Ms S's awareness of the CRIG framework improved during the study, as evident in the increased frequency of references in the third interview. Ms N's interview reflections on her lessons highlighted an increasing awareness of noticing structural thinking, her responses to how the students' approached the problems given, such as creating an increasing pattern, showed that she was noticing students' structural thinking when "in-the-moment" of the lesson.

The second research question was concerned with how the PSTs attend to structural thinking through their communications and the type of pedagogical practice employed to notice structural thinking. Ms N showed insight into the importance of students' understanding of the mathematical concepts and how the CRIG framework supported her pedagogical practice to achieve this. She engaged students in her lessons by using the CRIG

framework components of *Recognising patterns* in lesson 1 and 2 and *Identifying similarities and differences* in lesson 3. The PST's noticing of structural thinking through the CRIG framework linked the pedagogical practices designed to suit the lesson topic. Ms S's second lesson used *Generalising and reasoning* to choose informal units to estimate then measure lengths. Ms N third lesson used manipulative materials to identify similarities and differences between 3D objects.

Ms N's reflections identified her desire to understand her students' thinking. The CRIG framework provided opportunities for her to notice structural thinking through her communications. Examples of this were in lesson 1 when she asked questions when manipulating the blocks, so students were *Generalising and reasoning* a pattern, or in lesson 2 when she asked questions of students to *Recognise patterns* to add two-digit numbers, and in lesson 3 she encouraged students to *Identify similarities and differences* between 3D objects. Ms S and Ms N agreed that the CRIG framework had helped them formulate appropriate questions to ask students. They said that the questions they asked of the students and students' responses provided them with insights into structural thinking and opportunities to prompt students' thinking.

At the end of the study in the lesson 3 interview, Ms S said that she was aware of using the framework when communicating instructions, although previously she felt she had not. PSTs were communicating the CRIG framework in their lessons but were not always aware of doing so or identifying the individual component by name. Attention to structural thinking was occurring in the PSTs' pedagogical actions of modelling, giving instructions and communications.

PSTs' noticing structural thinking through the CRIG framework was not always explicit in their teaching. However, the PSTs noticed structural thinking through examples of the CRIG framework in the reflective practices in the interviews and reviewing segments of videoed lessons in the workshops. This reinforced Mason's (2011) notion that noticing is having an awareness of what one is attending to when teaching.

Mason (2004) approached noticing as a personal awareness of what one is attending to when teaching mathematics. Noticing structural thinking in this study required PSTs' awareness of when they were attending to the CRIG framework, which Mason (2002) called "acting-in-the-moment". Schoenfeld's (1992) description of metacognition in mathematical thinking and Mason's (2004) need for self-awareness of how one thinks when doing mathematics are of significance in this discussion as both PSTs were able to notice structural thinking through their actions and could think about it when reflecting on their teaching.

Conclusions and Further Research

The CRIG framework proved useful for the PSTs in this study. It provided them with a foundation for teaching mathematics that had a focus on developing conceptual and procedural understanding through mathematical structure. This framework could be helpful for all teachers as it supports the communication of mathematics to engage students through structural thinking. Further research of the CRIG framework could investigate how useful the framework is for students to develop structural thinking skills in mathematics.

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